PROBLEMS WITH PLURALS

Abstract. Plural quantification is thought to avoid Russell-type paradoxes. Yet, given certain modest metaphysical assumptions about propositions, plural quantification leads to Russell-type paradoxes. We can reject these metaphysical assumptions, but doing so leaves us with arguments against a variety of plausible claims about propositions. We formalize these arguments and show that there are no easy ways of escape.

1. Introduction

A Russell paradox is generated by one axiom schema for the membership relation $\in$:

set-comp. $\exists z \forall x(x \in z \leftrightarrow F(x))$

for any formula $F(x)$ open only in x. (Normally comprehension allows other free variables, but we don't need this.) For suppose we put the Russell formula $\sim(x \in x)$ for $F(x)$ in the schema. Then set-comp lets us formally prove a contradiction: there is an z such that $z \in z$ if and only if $\sim(z \in z)$.

Plural quantification is attractive because it promises to avoid Russell paradoxes. However, as McGee and Rayo (2000) point out, plural quantification can actually lead back into Russell paradoxes, given certain assumptions about propositions.[1] We offer a more generalized version of the path to paradox by showing that any theory that makes possible the construction of an appropriate *packaging* relation falls prey to a Russell paradox. We then give examples of widely-held metaphysical theories that require such a relation. One response to the argument is to drop certain "obvious" axioms of plural quantification. But we explain why doing that leads to other challenges. In the end, we find that the paradoxes that can result from plural quantification are more widely damaging and harder to tame than has been recognized. We also display formal requirements that

[1]See also Spencer 2012. Spencer's path to paradox is Cantorian in nature, whereas McGee and Rayo's is Russellian. We are here interested in the Russellian path, though we pay attention to the similarities. The path we take also differs from the paths to paradox marked out by Grim (1991, 1993). Those paths are expressed in terms of universal individual quantification, not plural quantification, and they require assumptions about aboutness and classes that are unnecessary for our arguments.

any metaphysical framework with a packing relation must meet if it is to have a chance of escaping self-contradiction.

We will first give the general paradox-generating setup, and then show how three families of metaphysical assumptions allow one to instantiate it. We'll close by assessing the aftermath of our arguments.

2. Packaging and paradox

In this section, we will mark out a pathway from plural quantification to a Russell paradox. The basic primitives of plural quantification are the quantifiers, $\exists xx$ ("there are some xs, such that") and $\forall xx$ ("for any xs") as well as the primitive $\prec$ ("is one of the") which can occur in the context $a \prec xx$. These primitives allow us to construct the plural analogues of quantifier introduction and elimination rules.

We may begin our journey toward paradox with the following axiom schema:

PL-COMP. $\exists x F(x) \rightarrow \exists xx \forall y (y \prec xx \leftrightarrow F(y))$

for any formula $F(x)$ open only in x. (Again, normally comprehension allows other free variables, but we don't need this.) PL-COMP says that if there is a formula that has a satisfier, then there are some things, the xs, which comprise all and only those things that satisfy it. More simply: if there are *any* Fs, then there are *the* Fs. It is worth noting that PL-COMP makes plural quantification useful in contexts where one might be suspicious of the use of sets, such as when talking about *all sets* or other things that can't be grouped into a set according to modern set theory.

We'll reach a paradox from here only if we have an appropriate *packaging* formula. The packaging formula is a formula $P(x, yy)$ open only in a singular variable x and a plural variable yy. Intuitively, $P(x, yy)$ will say x packages the ys. The thought is that x is an entity that encapsulates the ys according to the particular encapsulation method in the construction. We will consider a number of candidate constructions. An initial example of such encapsulation might be sets: one could take $P(x, yy)$ to say that x is the set of the ys, i.e., $\forall z(z \in x \leftrightarrow z \prec yy)$. The pluralist will not use sets for packaging, of course, for doing so offers no advantage over non-plural, set theoretic quantification.

Plus, to avoid a Russell paradox, the pluralist who accepts PL-COMP would need to deny that every plurality has a set.

Nevertheless, for a well-constructed packaging formula, a pluralist might still have reasons to accept a packaging axiom that says that each plurality has a package:

PACK. $\forall xx \exists u P(u, xx)$.

(Here, for a formula $F(x_1, \ldots, x_n)$, where some of these might be plural variables, we write $F(y_1, \ldots, y_n)$ for the result of respectively substituting $y_1, \ldots, y_n$ for the free instances of $x_1, \ldots, x_n$.)

Moreover, a package could be thought to contain exactly one plurality:

UNIQ. $\forall u \forall xx \forall yy((P(u, xx) \wedge P(u, yy)) \rightarrow xx = yy)$

where $xx = yy$ abbreviates $\forall u(z \prec xx \leftrightarrow z \prec yy)$. Note that although each package contains exactly one plurality, a plurality may be contained in more than one package.

And, finally, one might suppose that not everything is a package:

NONPACK. $\exists x(\sim \text{Package}(x))$

where $\text{Package}(x)$ abbreviates $\exists yy(P(x, yy))$. When packages are abstracta, then $P(x, yy)$ cannot hold for a concrete individual x such as the reader, and so NONPACK is true.

Then PL-COMP, PACK and UNIQ together let one prove:

PACK-COMP. $\exists x F(x) \rightarrow \exists z \forall x(x \lhd z \leftrightarrow F(x))$

for any formula $F(x)$ open only in x, where $x \lhd y$ abbreviates $\exists zz(P(y, zz) \wedge x \prec zz)$. (The context will make clear which packaging formula is used in defining $x \lhd y$. If packaging is done in terms of sets, then $\lhd$ will be the membership relation $\in$.)

We shall give the details of the proof shortly. But first, let's see why this matters.

Let $R(x)$ abbreviate the Russell-type formula $\sim(x \lhd x)$. Plugging this formula into PACK-COMP gives us a Russellian contradiction, *if* $\exists x(R(x))$. Yet, we can show that $\exists x(R(x))$ is just a consequence of NONPACK as follows:

1	$\exists x(\sim \text{Package}(x))$	nonpack
2	$a \quad \sim \text{Package}(a)$	
3	$a \in a$	
4	$\exists zz(P(a, zz) \wedge a \prec zz)$	def, 3
5	$bb \quad P(a, bb) \wedge a \prec bb)$	
6	$P(a, bb)$	$\wedge$-elim, 5
7	$\exists yyP(a, yy)$	$\exists$-intro, 6
8	$\exists yyP(a, yy)$	$\exists$-elim, 4, 5–7
9	$\text{Package}(a)$	def, 8
10	$\perp$	$\perp$-intro, 2, 9
11	$\sim(a \lhd a)$	$\sim$-intro, 3–10
12	$\exists x(\sim(x \lhd x))$	$\exists$-intro, 11
13	$\exists x(\sim(x \lhd x))$	$\exists$-elim, 1, 2–12
14	$\exists x(R(x))$	def, 13

We can now go on to formally derive a contradiction from PACK-COMP as follows:

15	$\exists xR(x) \to \exists z\forall x(x \lhd z \leftrightarrow R(x))$	pack-comp
16	$\exists z\forall x(x \lhd z \leftrightarrow R(x))$	$\to$-elim, 14, 15
17	$c \quad \forall x(x \lhd c \leftrightarrow R(x))$	
18	$c \lhd c \leftrightarrow R(c)$	$\forall$-elim, 17
19	$c \lhd c \leftrightarrow \sim(c \in c)$	def, 18
20	$c \lhd c$	
21	$\sim(c \lhd c)$	$\leftrightarrow$-elim, 19, 20
22	$\perp$	$\perp$-intro, 20, 21
23	$\sim(c \lhd c)$	$\sim$-intro, 20–22
24	$c \lhd c$	$\leftrightarrow$-elim, 19, 23
25	$\perp$	$\perp$-intro, 23, 2
26	$\perp$	$\exists$-elim, 16, 17—25

It remains to give the proof of PACK-COMP from PL-COMP, PACK and UNIQ. That proof is given in the Appendix. (Note that our proofs above and in the Appendix are intuitionistically-valid: double-negation elimination is never used.)

Pluralists must therefore work with a packaging formula satisfying other axioms or go without packaging. These options, however, have surprising consequences for our understanding of propositions, as we shall see next, since there are plausible constructions of packages that do satisfy the above axioms.

3. Metaphysics

We now offer some constructions of P on which PACK, UNIQ and NONPACK (or slight variants) are plausible. These constructions are built from certain reasonable assumptions about propositions. We thus show that plausible packages for plural quantification cause trouble for various claims about propositions. In doing so, we illustrate certain formal requirements that must be met by any metaphysical framework that includes packaging (such as theories of facts, propositions, states of affairs, collections, and so on). Unfortunately, many current theories fail to meet those requirements.[2] Throughout the arguments, we will be talking about propositions that allow for hyperintensional distinctions (as opposed to Lewis's (1986) propositions considered as sets of worlds): so, for example, the proposition that every dog is a dog is distinct from the proposition that Fermat's Last Theorem is true.

3.1. **Plural subjects of propositions.** Suppose, first, that propositions are abstract entities that don't depend for their existence upon particular concrete arrangements of (say) ink or sound-waves. Then it is plausible that for any ys there is a plural *de re* existential proposition claiming precisely the existence of these ys.[3] Where English has a referring expression "rs" for the ys, say "Jim and Bob" or "the actual world's dogs", this proposition can be expressed by "rs exist". But even when English lacks a corresponding referring expression, it is plausible that there will still be such propositions—assuming propositions are irreducible to sentence tokens.

But that's enough to land us in a Russell paradox. For with abstract propositions in hand, we may let $P(x, yy)$ say that x is a *de re* existential proposition claiming precisely the existence of the

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[2]Rosen (1995) applies an informal version of the Russellian "plurals" paradox to Armstrong's theory of states of affairs. We aim to show that the paradox, in the abstracted form we've presented, affects many other frameworks, too.

[3]Spencer (2012) endorses this premise in his Cantorian-style argument against an instance of PL-COMP (i.e., the instance where $F =$ 'x exists'). In a moment, we will consider some ramifications of rejecting PL-COMP.

ys.[4] And in that case, PACK and UNIQ will be true. For: (i) for any ys, we've assumed there is a *de re* existential proposition x claiming precisely their existence, and (ii) any zs that x precisely claims the existence of are the same as the ys. Moreover, the truth of NONPACK is undeniable: there exists a non-proposition—for instance the reader. Thus, by the proof we gave in the previous section, a contradiction results.

One reply is to deny that propositions are independent abstract objects. But even if propositions are concrete things such as sentence tokens or dependent on sentence tokens, there is another route to paradox using PL-COMP. First, let's give it informally. Consider all and only the propositions that don't ascribe existence to themselves. There are those propositions—by PL-COMP. And there is the proposition that they exist. I just stated it. Yet that proposition—the proposition that there are the propositions that don't ascribe existence to themselves—ascribes existence to *itself* if and only if it doesn't. We've again landed in contradiction.

Let us make the above argument more explicit. We stipulate that "$\exists! xx F(xx)$" means "There are unique xs such that $F(x\text{s})$", which in turn means "$\exists xx F(xx) \wedge (yy)(F(yy) \rightarrow yy = xx)$", where '$=$' is understood as before. Now suppose $F(x)$ is a formula in our language open only in x that has at least one satisfier. Then, $\exists x(F(x))$. And thus from PL-COMP we have: there are ys such that for all x, $F(x)$ iff x is one of the ys. Therefore, there are unique ys such that for all x, $F(x)$ iff x is one of the ys. Now, consider them. They exist. This token of "They exist" expresses a proposition that is *de rebus* about ys such that $F(x)$ iff x is one of the ys. In general, therefore, for any formula $F(x)$ open in one variable we can express, if there is a plurality of satisfiers, then there is, or can be, a sentence, and hence a proposition (perhaps a non-abstract one), about all and only the satisfiers. Now let $F(x) =$ 'x is a proposition that is not *de rebus* about objects that include x', and the contradiction results.[5]

We should add that there is nothing particularly special about existential propositions. We could run the construction instead in terms of plural predicative propositions in two ways. First way: fix

[4]This packaging is implied by Pruss's suggested reduction of collections to propositions (2011, p. 161).

[5]You might notice that the argument requires a semantic assumption: that for any formula open in one variable that is uniquely satisfied, we can successfully stipulate a referring term "D" that designates the satisfiers. We might compare the resulting paradox with the following non-plural, Grelling–Nelson paradox: 'non-self-describing' describes itself iff it doesn't. Perhaps a successful solution to the semantic paradox would provide a reason to deny the semantic premise in this formulation of our metaphysics paradox. We leave that open.

a property or relation Q, say *concreteness* or *mutual spatiotemporal unrelatedness* or even *acting together*[6], and say that for any ys there is a plural predicative proposition that precisely attributes Q to the ys (and the attributions don't have to be correct). Alternatively, one could more generally say that for any ys there is a plural predicative proposition that precisely attributes *some* property to the ys. Either way we can generate packages. The second way will presumably result in an infinite number of packages for any given plurality, but the arguments that lead to paradox did not assume that a plurality has only one package.

3.2. **Unrestricted fusion and singular *de re* propositions.** Within the context of something like classical mereology, assume unrestricted fusion, in plural formulation:

SUM. $\forall xx[\exists y(y \prec xx) \to \exists z\Sigma(z, xx)]$

where $\Sigma(z, xx)$ is your favorite formulation of the claim that z is a mereological sum or fusion of the xs. For instance:

$$\forall w(z \sqcap w \leftrightarrow \exists x(x \prec xx \wedge x \sqcap w)),$$

(cf. Varzi 2009, P.12$_\xi$), where $\sqcap$ is the overlap predicate.[7] It won't do to package up the xs just as their fusion, for different pluralities will have the same package, which will violate PACK: for instance, the plurality a and $b+c$ ("$b+c$" denotes the fusion of b and c) and the plurality $a+b$ and c have the same fusion. We need something a little less direct.

Let $E(x,y)$ say that x is a proposition that *de re* claims of y that y exists. Now we define $P(x, yy)$ to claim that x is a fusion of propositions making *de re* existential claims about individual ys, at least one such proposition for each of the ys. Namely $P(x, yy)$ says:

$$\exists zz[\Sigma(x, zz) \wedge \forall u(u \prec zz \to \exists v(E(u,v) \wedge v \prec yy)) \\ \wedge \forall v(v \prec yy \to \exists u(E(u,v) \wedge u \prec zz))].$$

[6]Notice that these examples apply differently to a plurality. The xs are concrete just in case each of the xs is concrete. The xs are mutually spatiotemporally unrelated just in case any two of the x are spatiotemporally unrelated. On the other hand acting together does not admit of any such easy analysis in terms of individuals.

[7]Plural quantification is not essential to this version of the paradox. We could instead formulate the SUM axiom as a schema: $\exists x(F(x)) \to \exists z\Sigma_F(z)$, for any formula $F(x)$ open only in x. where $\Sigma_F(z)$ says that z is the fusion of all satisfiers of $F(x)$—e.g., formulated as, $\forall w(z \sqcap w \leftrightarrow \exists x(F(x) \wedge x \sqcap w))$.

For instance, if the ys are Bill and Ted, and x is the fusion of the proposition that Bill exists with the proposition that Ted exists, then $P(x, yy)$.

An advantage of the present approach is that it only requires singular *de re* subjects,[8] not the plural ones in the previous construction.

Once again the truth of NONPACK is obvious: not everything is a fusion of propositions. What about PACK and UNIQ? Well, for any ys, there will be a plurality of those zs that are propositions *de re* claiming of individual ys that they exist. And by SUM, there will be a fusion f of those zs. This f packages the ys, and so we get PACK.

Moreover, it is very plausible that, as UNIQ says, if f packages the y, then a singular existential proposition is a part of f if and only if it is a singular existential proposition claiming the existence of one of the ys. For while singular existential propositions may have structure, plausibly they lack *mereological* structure. The most plausible theory where singular propositions would have classical mereological structure is one where the proposition that attributes property Q to x is the fusion of x and Q. But that theory must be false, because if Q and R are distinct properties, then the proposition that attributes Q to R differs from the proposition that attributes R to Q, while the two corresponding fusions will be identical. (Of course, there are plausible theories on which singular propositions have non-mereological structure. See, for example, Rasmussen Forthcoming. And for a radically non-classical mereological account, see Tillman and Fowler, 2011.)

And if singular propositions lack mereological structure, then the only way a singular existential proposition could be a part of a fusion of singular existential propositions would be by being one of these propositions. This claim shows that if z is one of the xs, then some singular existential proposition claiming the existence of z is a part of f. But then that proposition will also affirm the existence of one of the ys. And so every one of the xs is one of the ys. And the converse goes the same way. Thus, the xs are the ys, which is what UNIQ needs.

As in the previous method, if one doesn't like singular existential propositions here, one can have $E(x, y)$ say that x attributes concreteness to y, or that x attributes some property or other to y, etc.

[8] For discussions of this approach, see Hudson (2006) and Spencer (2006).

This version of the paradox basically follows Lewis's construction of set theory out of a singleton function and unrestricted fusion (Lewis 1991). Lewis escapes the Russell paradox by insisting that some objects (the proper classes) do not have a singleton. The point of our version is to note that a non-Lewisian *de re* singular propositions appears to provide a singleton for any object. Lewis is in the unwelcome position of having to deny that for every object there is a *de re* structured (in his terminology–see Lewis 1986, p. 57) proposition about it. It is very odd to suppose some objects are subjects of *de re* propositions and others are not.

Still, you might wonder if there is a way to avoid the paradox by pursuing a mereological theory of propositions, for mereological theories typically lead a denial of UNIQ.[9] We will consider one such theory discussed by Hudson (2006). His solution, which when adapted to our setting, says that some singular existential propositions are fusions of other singular propositions.[10] Specifically, if c is a fusion of a and b, then the proposition $\langle c \text{ exists}\rangle$ is a fusion of $\langle a \text{ exists}\rangle$ and $\langle b \text{ exists}\rangle$. This leads to the denial of UNIQ. Hence, we can escape the above formulation of the paradox if we accept this mereological theory of propositions.

Unfortunately, we still aren't home free, for we can tweak the packaging. Instead of packaging the ys into a fusion of singular existential propositions, one for each of the ys, we may instead package the ys into a fusion of negations of singular existential propositions, one for each of the ys. Alternatively, we may package the ys into a fusion of propositions, each of which predicates a non-compositional predicate like simplicity of one of the ys.[11] On either packaging, UNIQ is plausible, even granting Hudson's mereological theory. So, these packages land us back in paradox. Perhaps there is a way to further develop Hudson's theory so that we can escape paradox even on the above packagings. But if so, it's far from obvious what that is. We conclude, then, that these versions of the paradox pose an unsolved challenge for mereological theories of propositions.

3.3. **Unrestricted conjunction or disjunction.** Suppose that any plurality of propositions has a proposition that is a conjunction of the propositions in the plurality. (If there is only one proposition

[9]For example, if x is the sum of a and b, then x is also identical to the sum of x, a and b.

[10]We are grateful to an anonymous reader for the adaptation.

[11]If P is a compositional predicate, i.e., one such that the fusion of the ys satisfies P if and only if each of the ys satisfies P, then someone who finds Hudson's like of thought plausible may think that $\langle P(c)\rangle$ is a fusion of $\langle P(a)\rangle$ and $\langle P(b)\rangle$ when c is a fusion of a and b.

in the plurality, we stipulate that that item trivially counts as a conjunction of the propositions in the plurality.) Then instead of doing packaging with fusions of singular *de re* propositions as in the previous section, we can do it with conjunctions (or disjunctions, respectively) of such propositions. In fact, we can even do this with the same definition of $P(x, zz)$ as long as we reinterpret $\Sigma(x, zz)$ to say that x is a conjunction (respectively, disjunction) of the zs. UNIQ will then require the very plausible claim that if x is a conjunction (disjunction) of ys that are singular *de re* propositions of the right type (say, existential), then all the singular *de re* propositions that are conjuncts of x are among the ys.

4. Ways out

We now have several families of routes to paradox. All require plural comprehension or unrestricted fusion, and the existence of propositions. The route of Section 3.1 requires a further controversial condition that for any plurality, there is a proposition (existential or predicative) precisely about that plurality. The route of 3.2, instead, requires unrestricted fusions even of abstract objects, and requires that singular or negative propositions not be mereologically composed. The route of 3.3 requires unrestricted conjunctions or disjunctions.

4.1. **No plural quantification.** A way to cut most of the problems off at the source is to deny plural comprehension (and also deny SUM-SCHEMA).[12] But unless we want to forego plural quantification entirely, we will want a replacement. In the case of set theory, the replacement for comprehension was a number of axioms most notably including separation: the claim that for any set a and formula $F(x)$, there is the set consisting of all the elements of a satisfying $F(x)$ (unlike comprehension, this doesn't allow a set to be built from scratch). We could likewise have the schema:

PL-SEP. $\forall yy \exists x(x \prec yy \wedge F(x)) \to \exists xx \forall y[y \prec xx \leftrightarrow (y \prec yy \wedge F(y))]$.

But then we couldn't correctly affirm that the propositions exist, whether propositions are understood as concrete or abstract. For suppose there is a plurality of propositions. Then we can adapt our arguments to use PL-SEP in place of PL-COMP as long as our Russell formula is replaced with $x \prec pp \wedge x \notin x$, where the ps are the propositions. So this way out has a cost: although one could

[12]This is the way out Spencer (2012) recommends in response to his Cantorian "plurals" paradox.

still hold on to the claim that there is *a* proposition, i.e., $\exists x(\text{Prop}(x))$, one could no longer hold that there are *the* propositions, i.e., $\exists xx(\forall y(y \prec xx \leftrightarrow \text{Prop}(y)))$. This is not very plausible. After all, "The propositions are either spatiotemporally unrelated or spatiotemporally related" seems to express a truth and to have as its subject a plurality.

Moreover, PL-SEP is nowhere nearly a sufficient replacement for PL-COMP. Just as in Zermelo-Fraenkel set theory other axioms had to be added beyond separation to make up for the lack of comprehension, here too other axioms will be needed. After all, by itself, PL-SEP is compatible with there not being any pluralities at all. We might, for instance, want to have axioms like:

PL-SING. $\forall x \exists yy \forall z(z \prec zz \leftrightarrow z = x)$

PL-UNION. $\forall xx \forall yy \exists zz(u \prec zz \leftrightarrow (u \prec xx \vee u \prec yy))$

Together, these imply that for any finite sequence of objects $a_1, a_2, \ldots, a_n$, there is a plurality of a_1, a_2, ... and a_n. The concern with this approach is that one is recreating something that is very much like set theory over again, thereby stripping plural quantification of its principal advantage over non-plural set-theoretic quantification. Plus, we still can't affirm that the propositions exist, which is a problem. So, giving up PL-COMP leads to other challenges.

4.2. **No propositions.** A metaphysically radical way out is to deny that there are any propositions at all (whether concrete or abstract), i.e., to deny $\exists x(\text{Prop}(x))$. This leads to difficulties in accounting for content and the objects of propositional attitudes, and is indeed radical. There are, of course, various ways to try to mitigate the harshness of this move, such as by motivating fictionalism about propositions (see Balaguer 1998 and 2010) or finding an appropriate nominalist or conceptualist replacement (see, for example, Pruss 2011, pp. 274–275; cf. Alston 1986).

4.3. **No infinite propositions and no unrestricted fusions.** One could try for a more finely-grained response to the methods. A common requirement of Sections 3.1 and 3.3 is the existence of certain "infinite propositions", whether infinite because they have an infinite plurality *de rebus* as their subject or because they are infinite conjunctions (or disjunctions). By denying the existence of such infinite propositions, one undercuts all the paradoxes except those based on unrestricted fusion. For this strategy not to be *ad hoc*, it seems we should simply deny all infinite propositions.

We can also get out of the fusion-based method (3.2) by noting that the hypothesis of unrestricted fusions is perhaps the most controversial of our technical assumptions, and there are independent intuitive reasons to deny that assumption. For instance, one might think that everything is either abstract or concrete but not both, and that abstract things do not have concrete parts and concrete things do not have abstract parts. If one thinks this, then there won't be a fusion of an abstract and a concrete object. Or one might think that mereological concepts of overlap depend on spatial relationships, in which case abstract objects won't have fusions—and it is precisely abstract objects that we need to exclude from fusions.

The weak part of this combination approach is that there in fact is good reason to think there could be infinite propositions. Our best guide to the existence of a proposition of a certain sort is the possibility of a sentence that would express that proposition. But the finite sentence "The propositions (or, the actually existing propositions) are mutually spatiotemporally unrelated" seems to be an English sentence with an infinite plurality *de rebus* as a subject, and it attributes spatiotemporal unrelatedness to that plurality. Thus it seems precisely to express a proposition of the sort we this approach says to be nonexistent. Moreover, one could perhaps have beings whose method of linguistic expression made infinite conjunctions possible. For instance, perhaps, they could just utter an infinite English sentence in a finite amount of time as a supertask. Or perhaps one could have a being that could utter each of infinitely many conjuncts at once, with that being understood to be a conjunction of them, with each conjunct being uttered at a different range of frequencies (this will work better with electromagnetic than sound waves). So cutting out infinite propositions is not cheap.

And even an *ad hoc* piecemeal approach, which allows there to be some infinite propositions but denies that for all xs there is a proposition *de rebus* about the xs, and which allows there to be some infinite conjunctions but denies that all infinite pluralities have a conjunction, runs into the difficulty that we seem to be able to form sentences that seem to express precisely the forbidden propositions. As we saw in Section 3.1, for any formula $F(x)$ open only in x that has something satisfying it and that can be expressed in English, we can form an English sentence that is *de rebus* about the ys that satisfy $F(x)$. We do this by first introducing a context where such ys are contextually relevant, say by saying "There exist unique ys such that something is one of the ys if and only if it satisfies $F(x)$", and then using a plural pronoun as the subject of a sentence, say "They exist", where the context makes it clear that the pronoun refers to such ys.

4.4. **Assessment.** The above approaches appear to exhaust the viable options. If plural comprehension is rejected, we have the way out of Section 4.1. If we accept plural comprehension, then either we must deny that there are propositions (4.2) or deny more specifically that there are the special kinds of propositions needed by the two routes to paradox that use infinite propositions (3.1 and 3.3).

The two most principled ways out appear to be to deny plural comprehension (4.1) or to deny that there are any propositions at all (4.2). The former is the less metaphysically radical approach, especially if we introduce plural separation and other axioms mirroring those of set theory, but it does lose us some of the advantages of using plural quantification instead of sets or classes. The other approaches each require denying the existence of some propositions that seem to be expressed by contentful English sentences, while accepting that there are such things as propositions. The results in any case are perplexing and instructive.[13]

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[13]Our arguments can be viewed as a development of Russell's own "propositions" paradox given at the end of his *Principles of Mathematics*. Russell (1913, p. 527) confesses to have no solution—not even in terms of his theory of types. He writes, "What the complete solution of the difficulty may be, I have not succeeded in discovering; but as it affects the very foundations of reasoning, I earnestly commend the study of it to the attention of all students of logic." The situation remains.

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Appendix: Proof of pack-comp from pl-comp, pack and uniq

The following proof schema, valid for any formula $F(x)$ open only in x, is also intuitionistically valid.

1		$\exists x F(x)$	
2		$\exists x F(x) \to \exists xx \forall y(y \prec xx \leftrightarrow F(y))$	pl-comp
3		$\exists xx \forall y(y \prec xx \leftrightarrow F(y))$	→-elim, 1, 2
4	aa	$\forall y(y \prec aa \leftrightarrow F(y))$	
5		$\forall xx \exists u P(u, xx)$	pack
6		$\exists u(P(u, aa)$	∀-elim, 5
7	b	$P(b, aa)$	
8	c	$c \lhd b$	
9		$c \prec aa \leftrightarrow F(c)$	∀-elim, 4
10		$\exists zz(P(b, zz) \wedge c \prec zz)$	def, 8
11	dd	$P(b, dd) \wedge c \prec dd$	
12		$P(b, dd)$	∧-elim, 11
13		$\forall u \forall xx \forall yy((P(u, xx) \wedge P(u, yy)) \to xx = yy)$	uniq
14		$(P(b, aa) \wedge P(b, dd)) \to aa = dd$	∀-elim, 13
15		$P(b, aa) \wedge P(b, dd)$	∧-intro, 7, 12
16		$aa = dd$	→-elim, 14, 15
17		$\forall z(z \prec aa \leftrightarrow z \prec dd)$	def, 16
18		$c \prec aa \leftrightarrow c \prec dd$	∀-elim, 17
19		$c \prec dd$	∧-elim, 11
20		$c \prec aa$	↔-elim, 18, 19
21		$F(c)$	↔-elim, 9, 20
22		$F(c)$	∃-elim, 10, 11–21
23		$F(c)$	
24		$c \prec aa$	↔-elim, 9, 23
25		$P(b, aa) \wedge c \prec aa$	∧-intro, 7, 24
26		$\exists zz(P(b, zz) \wedge c \prec zz)$	∃-intro, 25
27		$c \lhd b$	def, 26
28		$c \lhd b \leftrightarrow F(c)$	↔-intro, 8–22, 23–27
29		$\forall x(x \lhd b \leftrightarrow F(x))$	∀-intro, 8–28
30		$\exists z \forall x(x \lhd z \leftrightarrow F(x))$	∃-intro, 29
31		$\exists z \forall x(x \lhd z \leftrightarrow F(x))$	∃-elim, 6, 7–30
32		$\exists z \forall x(x \lhd z \leftrightarrow F(x))$	∃-elim, 3, 4–31
33		$\exists x F(x) \to \exists z \forall x(x \lhd z \leftrightarrow F(x))$	→-intro, 1, 2–32